

THE SEQUENCE SPACE bv AND SOME APPLICATIONS

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ABSTRACT. In this work, we give well-known results related to some properties, dual spaces and matrix transformations of the sequence space bv and introduce the matrix domain of space bv with arbitrary triangle matrix A . Afterward, we choose the matrix A as Cesàro mean of order one, generalized weighted mean and Riesz mean and compute α -, β -, γ -duals of these spaces. And also, we characterize the matrix classes of the new spaces.

1. INTRODUCTION

The set of all sequences denotes with $\omega := \mathbb{C}^{\mathbb{N}} := \{x = (x_k) : x : \mathbb{N} \rightarrow \mathbb{C}, k \rightarrow x_k := x(k)\}$, where \mathbb{C} denotes the complex field and $\mathbb{N} = \{0, 1, 2, \dots\}$. Each linear subspace of ω (with the induced addition and scalar multiplication) is called a *sequence space*. We will write ϕ, ℓ_∞, c and c_0 for the sets of all finite, bounded, convergent and null sequences, respectively. It obviously that these sets are subsets of ω .

A sequence, whose k -th term is x_k , is denoted by x or (x_k) . By e and $e^{(n)}$, $(n = 0, 1, 2, \dots)$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, 2, \dots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

A *coordinate space* (or K -space) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space X with a linear topology is called a K -space provided each of the maps $p_i : X \rightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A BK -space is a K -space, which is also a Banach space with continuous coordinate functionals $f_k(x) = x_k$, $(k = 1, 2, \dots)$. A K -space K is called an FK -space provided X is a complete linear metric space. An FK -space whose topology is normable is called a BK -space. A sequence (b_n) , $(n = 0, 1, 2, \dots)$ in a linear metric X is called a *Schauder basis* if for each $x \in X$ there exists a unique sequence (α_n) , $(n = 0, 1, 2, \dots)$ of scalars such that $x = \sum_{n=0}^{\infty} \alpha_n b_n$. An FK -space X is said to have AK property, if $\phi \subset X$ and $\{e^{(n)}\}$ is a basis for X and $\phi = \text{span}\{e^{(n)}\}$, the set of all finitely non-zero sequences.

The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) , and written as $x = \sum \alpha_k b_k$. An FK -space X is said to have AK property, if $\phi \subset X$ and $\{e^k\}$ is a basis for X , where e^k is a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$ and $\phi = \text{span}\{e^k\}$, the set of all finitely non-zero sequences.

Let X is a sequence space and A is an infinite matrix. The sequence space

$$(1.1) \quad X_A = \{x = (x_k) \in \omega : Ax \in X\}$$

is called the matrix domain of X which is a sequence space (for several examples of matrix domains, see [6] p. 49-176).

We write \mathcal{U} for the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$. For $u \in \mathcal{U}$, let $1/u = (1/u_k)$. Let $u, v \in \mathcal{U}$, (t_k) be a sequence of positive and write $T_n = \sum_{k=0}^n t_k$. Now, we define the difference matrix $\Delta = (\delta_{nk})$, the matrix $C = (c_{nk})$ of the Cesaro mean of order one, the *generalized weighted mean* or *factorable matrix* $G(u, v) = (g_{nk})$ and the matrix $R^t = (r_{nk}^t)$ of the Riesz mean by

$$(1.2) \quad \begin{aligned} \delta_{nk} &= \begin{cases} (-1)^{n-k} & , \quad (n-1 \leq k \leq n) \\ 0 & , \quad (0 \leq k < n-1 \text{ or } k > n) \end{cases} \\ c_{nk} &= \begin{cases} \frac{1}{n+1} & , \quad (0 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases} \\ g_{nk} &= \begin{cases} u_n v_k & , \quad (0 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases} \end{aligned}$$

$$r_{nk}^t = \begin{cases} t_k/T_n & , \quad (0 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$; where u_n depends only on n and v_k only on k .

In this work, we give well-known results related to some properties, dual spaces and matrix transformations of the sequence space bv and introduce the matrix domain of space bv with arbitrary triangle matrix A . Afterward, we choose the matrix A as Cesàro mean of order one, generalized weighted mean and Riesz mean and compute α -, β - $\beta\gamma$ -duals of these spaces. And also, we characterize the matrix classes of the spaces $bv(C)$, $bv(G)$, $bv(R)$.

2. WELL-KNOWN RESULTS

In this section, we will give some well-known results and will define a new form of the sequence space bv with arbitrary triangle A .

The space of all sequences of bounded variation defined by

$$bv = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} |x_k - x_{k-1}| < \infty \right\},$$

which is a BK -space under the norm $\|x\|_{bv} = |x_0| + \sum_{k=1}^{\infty} |x_k - x_{k-1}|$ for $x \in bv$. The space bv_0 denotes $bv_0 = bv \cap c_0$. It is clear that $bv = bv_0 + \{e\}$. Also the inclusions $\ell_1 \subset bv_0 \subset bv = bv_0 + \{e\} \subset c$ are strict.

Let X be a sequence space and Δ denotes the matrix as defined by (1.2). The matrix domain X_{Δ} for $X = \{\ell_{\infty}, c, c_0\}$ is called the *difference sequence spaces*, which was firstly defined and studied by Kizmaz [22]. If we choose $X = \ell_1$, the space $\ell_1(\Delta)$ is called *the space of all sequences of bounded variation* and denote by bv . In [8], Başar and Altay have defined the sequence space bv_p for $1 \leq p \leq \infty$ which consists of all sequences such that Δ -transforms of them are in ℓ_p and have studied several properties of these spaces. Basar and Altay[8] proved that the space bv_p is a BK -space with the norm $\|x\|_{bv_p} = \|\Delta x\|_{\ell_p}$ and linearly isomorphic to ℓ_p for $1 \leq p \leq \infty$. The inclusion relations for the space bv_p are given in [8] as below:

- (i) The inclusion $\ell_p \subset bv_p$ strictly holds for $1 \leq p \leq \infty$.
- (ii) Neither of the spaces bv_p and ℓ_{∞} includes the other one, where $1 < p < \infty$.
- (iii) If $1 \leq p < s$, then $bv_p \subset bv_s$.

Define a sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$ of elements of the space bv_p for every fixed $k \in \mathbb{N}$ by 0 for $n < k$ and 1 for $n \geq k$. Then the sequence $\{b_n^{(k)}\}_{n \in \mathbb{N}}$ is a Schauder basis for the space bv_p and every sequence $x \in bv_p$ has a unique representation $x = \sum_k (\Delta x)_k b^{(k)}$ for all $k \in \mathbb{N}$. The space bv_{∞} has no Schauder basis[8].

Let X and Y be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} , where $k, n \in \mathbb{N}$. Then, we say that A defines a *matrix mapping* from X into Y , and we denote it by writing $A : X \rightarrow Y$ if for every sequence $x = (x_k) \in X$. The sequence $Ax = \{(Ax)_n\}$, the A -transform of x , is in Y ; where

$$(2.1) \quad (Ax)_n = \sum_k a_{nk} x_k \quad \text{for each } n \in \mathbb{N}.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞ . By $(X : Y)$, we denote the class of all matrices A such that $A : X \rightarrow Y$. Thus, $A \in (X : Y)$ if and only if the series on the right side of (2.1) converges for each $n \in \mathbb{N}$ and each $x \in X$ and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y$ for all $x \in X$. A sequence x is said to be *A-summable to l* if Ax converges to l which is called the *A-limit* of x .

If $X, Y \subset \omega$ and z any sequence, we can write $z^{-1} * X = \{x = (x_k) \in \omega : xz \in X\}$ and $M(X, Y) = \bigcap_{x \in X} x^{-1} * Y$. If we choose $Y = \ell_1, cs, bs$, then we obtain the α -, β -, γ -duals of X , respectively as

$$\begin{aligned} X^{\alpha} &= M(X, \ell_1) = \{a = (a_k) \in \omega : ax = (a_k x_k) \in \ell_1 \text{ for all } x \in X\} \\ X^{\beta} &= M(X, cs) = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x \in X\} \\ X^{\gamma} &= M(X, bs) = \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs \text{ for all } x \in X\}. \end{aligned}$$

The α -, β -, γ -duals of classical sequence spaces are defined by

$$\begin{aligned} c_0^{\alpha} &= c^{\alpha} = \ell_{\infty}^{\alpha} = \ell_1, & \ell_1^{\alpha} &= \ell_{\infty}, & cs^{\alpha} &= bs^{\alpha} = bv^{\alpha} = bv_0^{\alpha} = \ell_1 \\ c_0^{\beta} &= c^{\beta} = \ell_{\infty}^{\beta} = \ell_1, & \ell_1^{\beta} &= \ell_{\infty}, & cs^{\beta} &= bv, & bs^{\beta} &= bv_0, & bv^{\beta} &= cs, & bv_0^{\beta} &= bs \\ c_0^{\gamma} &= c^{\gamma} = \ell_{\infty}^{\gamma} = \ell_1, & \ell_1^{\gamma} &= \ell_{\infty}, & cs^{\gamma} &= bs^{\gamma} = bv, & bv^{\gamma} &= bv_0^{\gamma} = bs. \end{aligned}$$

The *continuous dual* of X is the space of all continuous linear functionals on X and is denoted by X^* . The continuous dual X^* with the norm $\|\cdot\|^*$ defined by $\|f\|^* = \sup \{|f(x)| : \|x\| = 1\}$ for $f \in X^*$.

The spaces c^* and c_0^* are norm isomorphic with ℓ_1 . Also, $\|a\|_\beta = \|a\|_{\ell_1}$ for all $a \in \ell_\infty^\beta$.

The following theorem is the most important result in the theory of matrix transformations:

Theorem 2.1. *Matrix transformations between BK -spaces are continuous.*

In [5], Başar and Altay developed very useful tools for duals and matrix transformations of sequence spaces as below:

Theorem 2.2. [8, Lemma 5.3] *Let X, Y be any two sequence spaces, A be an infinite matrix and U a triangle matrix. Then, $A \in (X : Y_U)$ if and only if $UA \in (X : Y)$.*

Theorem 2.3. [5, Theorem 3.1] $B_\mu^U = (b_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and inverse of the triangle matrix $U = (u_{nk})$ by

$$b_{nk} = \sum_{j=k}^n a_j u_{jk}$$

for all $k, n \in \mathbb{N}$. Then,

$$\lambda_U^\beta = \{a = (a_k) \in \omega : B^U \in (\lambda : c)\}$$

and

$$\lambda_U^\gamma = \{a = (a_k) \in \omega : B^U \in (\lambda : \ell_\infty)\}.$$

The following theorem proved by Zeller [38]:

Theorem 2.4. *Let X be an FK -space whose topology is given by means of the seminorms $\{q_n\}_{n=1}^\infty$ and let A be an infinite matrix. Then X_A is an FK -space when topologized by*

$$\begin{aligned} x &\rightarrow |x_j| & (j = 1, 2, \dots) \\ x &\rightarrow \sup_n \left| \sum_{j=1}^n a_{ij} x_j \right| & (i = 1, 2, \dots) \\ x &\rightarrow q_n(Ax) & (n = 1, 2, \dots). \end{aligned}$$

Following theorems are given by Bennet [19].

Theorem 2.5. *An FK -space X contains ℓ_1 if and only if $\{e^{(j)} : j = 1, 2, \dots\}$ is a bounded subsets of X .*

Theorem 2.6. *Let A be a matrix and X an FK -space. Then A maps ℓ_1 into X if and only if the columns of A belong to X and form a bounded subset there.*

Theorem 2.7. *An FK -space contains $bv_0(bv)$ if and only if $(e \in E)$ and $\{\sum_{j=1}^n e^{(j)} : j = 1, 2, \dots\}$ is a bounded subsets of X .*

Theorem 2.8. *Let A be a matrix and X an FK -space. Then A maps bv_0 into X if and only if the columns of A belong to X and their partial sums form a bounded subset there.*

The important class of co-null FK -space was introduced by Snyder [35]: An FK -space X containing $\varphi \cup \{e\}$ is said to be *co-null* if $\sum_{j=1}^\infty e^j = e$ weakly in X .

Therefore, using the definition of co-null and Theorem 2.7, we have:

Corollary 2.9. [19] *Any co-null FK -space must contain bv .*

Theorem 2.10 (Theorem 4.3.2 in [37]). *Let $(X, \|\cdot\|)$ be a BK -space. Then X^β is a BK -space with $\|a\|_\beta = \sup \{\sup_n |\sum_{k=0}^n a_k x_k| : \|x\| = 1\}$.*

Theorem 2.11 (Theorem 7.2.9 in [37]). *The inclusion $X^\beta \subset X^*$ holds in the following sense: Let the $\hat{\cdot} : X^\beta \rightarrow X^*$ be defined by $\hat{a} = \hat{a} : X \rightarrow \mathbb{C}, (a \in X^\beta)$ where $\hat{a}(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in X$. Then $\hat{\cdot}$ is an isomorphism into X^* . If X has AK , then the map $\hat{\cdot}$ is onto X^* .*

Now, we define the space bv with a lower triangle matrix $A = (a_{nk})$ for all $k, n \in \mathbb{N}$ as below:

$$(2.2) \quad bv(A) = \{x = (x_k) \in \omega : Ax \in bv\}.$$

The space bv is BK -space with the norm $\|x\|_{bv} = \|\Delta x\|_{\ell_1}$ and A is a triangle matrix. Then, from Theorem 4.3.2 in [37], we say that the space $bv(A)$ is a BK -space with the norm $\|x\|_{bv(A)} = \|Ax\|_{bv}$.

If A is a triangle, then one can easily observe that the sequence X_A and X are linearly isomorphic, i.e., $X_A \cong X$. Therefore, the spaces bv and $bv(A)$ are norm isomorphic to the space ℓ_1 and bv , respectively.

Now we give some results:

Lemma 2.12. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

(i) $A \in (\ell_1 : \ell_\infty)$ if and only if

$$(2.3) \quad \sup_{k, n \in \mathbb{N}} |a_{nk}| < \infty.$$

(ii) $A \in (\ell_1 : c)$ if and only if (2.3) holds, and there are $\alpha_k \in \mathbb{C}$ such that

$$(2.4) \quad \lim_{n \rightarrow \infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}.$$

(iii) $A \in (\ell_1 : \ell_1)$ if and only if

$$(2.5) \quad \sup_{k \in \mathbb{N}} \sum_n |a_{nk}| < \infty.$$

3. APPLICATIONS

In this section, we give some results related to the space $bv(A)$ as choose well-known matrices instead of arbitrary matrix A .

3.1. The space $bv(C)$. We give the matrix $C_1 = (c_{nk})$ instead of the matrix $A = (a_{nk})$, in (2.2). Then, we obtain the space $bv(C)$ as below:

$$(3.1) \quad bv(C) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{k+1} \sum_{j=0}^k x_j - \frac{1}{k} \sum_{j=0}^{k-1} x_j \right| < \infty \right\}.$$

Using the notation (1.1), we can denote the space $bv(C)$ as $bv(C) = (bv)_{C_1} = (\ell_1)_{\Delta, C_1}$, where $\Phi = \phi_{nk} = \Delta.C_1$ defined by $\phi_{nk} = \frac{-1}{(n+1).n}$ ($0 \leq k < n$); $\phi_{nk} = \frac{1}{n}$ ($k = n$) and $\phi_{nk} = 0$ ($k > n$) for all n .

The Φ -transform of the sequence $x = (x_k)$ defined by

$$(3.2) \quad y_k = \frac{x_k}{k+1} - \frac{1}{k(k+1)} \sum_{j=0}^k x_j.$$

Theorem 3.1. *Define a sequence $t^{(k)} = \{t_n^{(k)}\}_{n \in \mathbb{N}}$ of elements of the space $bv(C)$ for every fixed $k \in \mathbb{N}$ by*

$$t_n^{(k)} = \begin{cases} (k+1) & , \quad (n = k) \\ 1 & , \quad (n < k) \\ 0 & , \quad (n > k) \end{cases}$$

Therefore, the sequence $\{t^{(k)}\}_{k \in \mathbb{N}}$ is a basis for the space $bv(C)$ and any $x \in bv(C)$ has a unique representation of the form

$$x = \sum_k (\Phi x)_k t^{(k)}.$$

This theorem can be proved as Theorem 3.1 in [8], we omit details.

Theorem 3.2. *The α -dual of the space $bv(C)$ is the set*

$$d_1 = \left\{ a = (a_k) \in \omega : \sup_{k \in \mathbb{N}} \sum_n |b_{nk}| < \infty \right\}$$

where the matrix $B = (b_{nk})$ is defined via the sequence $a = (a_n) \in \omega$ by $(n+1)a_n$ ($n = k$), a_k ($k < n$) and 0 ($k > n$).

Proof. Let $a = (a_n) \in \omega$. Using the relation (3.2), we obtain

$$(3.3) \quad a_n x_n = \sum_n \left| \sum_{k=0}^{n-1} y_k + (n+1)y_n \right| a_n = (By)_n$$

It follows from (3.3) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in bv(C)$ if and only if $By \in \ell_1$ whenever $y \in \ell_1$. We obtain that $a \in [bv(C)]^\alpha$ whenever $x \in bv(C)$ if and only if $B \in (\ell_1 : \ell_1)$. Therefore, we get by Lemma 2.12 (iii) with B instead of A that $a \in [bv(C)]^\alpha$ if and only if $\sup_{k \in \mathbb{N}} \sum_n |b_{nk}| < \infty$. This gives us the result that $[bv(C)]^\alpha = d_1$. \square

Theorem 3.3. Define the sets by

$$\begin{aligned} d_2 &= \left\{ a = (a_k) \in \omega : \lim_n d_{nk} \text{ exists for each } k \in \mathbb{N} \right\} \\ d_3 &= \left\{ a = (a_k) \in \omega : \sup_k \sum_n |d_{nk}| < \infty \right\} \end{aligned}$$

where the matrix $D = (d_{nk})$ is defined via the sequence $a = (a_n) \in \omega$ by $(n+1)a_n \quad (n=k), \sum_{j=k}^n a_j \quad (k > n)$ and $0 \quad (k < n)$. Then, $[bv(C)]^\beta = d_2 \cap d_3$.

Proof. Consider the equation

$$(3.4) \quad \sum_{k=0}^n a_k x_k = \sum_{k=0}^n a_k \left[\sum_{j=0}^{k-1} y_j + (k+1)y_k \right] = (Dy)_n \quad (n \in \mathbb{N}).$$

Therefore, we deduce from Lemma 2.12 (ii) with (3.4) that $ax = (a_n x_n) \in cs$ whenever $x \in bv(C)$ if and only if $Dy \in c$ whenever $y \in \ell_1$. From (2.3) and (2.4), we have

$$\lim_n d_{nk} = \alpha_k \quad \text{and} \quad \sup_k \sum_n |d_{nk}| < \infty$$

which shows that $[bv(C)]^\beta = d_2 \cap d_3$. \square

Theorem 3.4. $[bv(C)]^\gamma = d_3$.

Proof. We obtain from Lemma 2.12 (i) with (3.4) that $ax = (a_n x_n) \in bs$ whenever $x \in bv(C)$ if and only if $Dy \in \ell_\infty$ whenever $y \in \ell_1$. Then, we see from (2.3) that $[bv(C)]^\gamma = d_3$. \square

In this subsection to use, we define the matrices for brevity that

$$\tilde{a}_{nk} = \sum_{j=0}^{k-1} a_{nj} + (k+1)a_{nk} \quad \text{and} \quad \tilde{b}_{nk} = \frac{a_{nk}}{n+1} - \frac{1}{n(n+1)} \sum_{j=0}^n a_{jk}$$

for all $k, n \in \mathbb{N}$.

Theorem 3.5. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation

$$(3.5) \quad e_{nk} = \tilde{a}_{nk}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (bv(C) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [bv(C)]^\beta$ for all $n \in \mathbb{N}$ and $E \in (\ell_1 : Y)$.

Proof. Let Y be any given sequence space. Suppose that (3.5) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces $bv(C)$ and ℓ_1 are norm isomorphic.

Let $A \in (bv(C) : Y)$ and take any $y = (y_k) \in \ell_1$. Then ΦE exists and $\{a_{nk}\}_{k \in \mathbb{N}} \in \{bv(C)\}^\beta$ which yields that (3.5) is necessary and $\{e_{nk}\}_{k \in \mathbb{N}} \in (\ell_1)^\beta$ for each $n \in \mathbb{N}$. Hence, Ey exists for each $y \in \ell_1$ and thus

$$\sum_k e_{nk} y_k = \sum_k a_{nk} x_k \quad \text{for all } n \in \mathbb{N},$$

we obtain that $Ey = Ax$ which leads us to the consequence $E \in (\ell_1 : Y)$.

Conversely, let $\{a_{nk}\}_{k \in \mathbb{N}} \in \{bv(C)\}^\beta$ for each $n \in \mathbb{N}$ and $E \in (\ell_1 : Y)$ hold, and take any $x = (x_k) \in bv(C)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^m a_{nk} x_k = \sum_{k=0}^m a_{nk} \left[\sum_{j=0}^{k-1} y_j + (k+1)y_k \right] = \sum_{k=0}^m \left(\sum_{j=k}^{m-1} a_{nj} y_j \right) + (m+1)a_{nm} y_m \quad \text{for all } m, n \in \mathbb{N}$$

as $m \rightarrow \infty$ that $Ax = Ey$ and this shows that $E \in (\ell_1 : Y)$. This completes the proof. \square

Theorem 3.6. Suppose that the entries of the infinite matrices $B = (\tilde{b}_{nk})$ and $F = (f_{nk})$ are connected with the relation

$$(3.6) \quad f_{nk} = \tilde{b}_{nk}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $B \in (Y : bv(C))$ if and only if $F \in (Y : \ell_1)$.

Proof. Let $z = (z_k) \in Y$ and consider the following equality

$$\sum_{k=0}^m \tilde{b}_{nk} z_k = \sum_{k=0}^m \left[\frac{a_{nk}}{n+1} - \frac{1}{n(n+1)} \sum_{j=0}^{n-1} a_{jk} \right] z_k \quad \text{for all } m, n \in \mathbb{N}$$

which yields that as $m \rightarrow \infty$ that $(Fz)_n = \{\Phi(Bz)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Bz \in bv(C)$ whenever $z \in Y$ if and only if $Fz \in \ell_1$ whenever $z \in Y$. \square

3.2. The space $bv(G)$. In (2.2), if we choose the matrix $G = G(u, v) = (g_{nk})$ instead of the matrix $A = (a_{nk})$, then, we obtain the space $bv(G)$ as below:

$$(3.7) \quad bv(G) = \left\{ x = (x_k) \in \omega : \sum_k \left| \sum_{j=0}^k u_k v_j x_j - \sum_{j=0}^{k-1} u_{k-1} v_j x_j \right| < \infty \right\}.$$

Using the notation (1.1), we can denote the space $bv(G)$ as $bv(G) = (bv)_{G(u,v)} = (\ell_1)_{\Delta.G(u,v)}$, where $\Gamma = \gamma_{nk} = \Delta.G(u, v)$ defined by $\gamma_{nk} = (u_n - u_{n-1})v_k$ ($1 \leq k < n$); $\gamma_{nk} = u_n v_n$ ($k = n$) and $\gamma_{nk} = 0$ ($k > n$) for all n .

The Γ -transform of the sequence $x = (x_k)$ defined by

$$(3.8) \quad y_k = \sum_{j=0}^{k-1} (u_k - u_{k-1}) v_j x_j + u_k v_k x_k.$$

$$(3.9)$$

The Theorem 3.7 and Theorem 3.8 can be proved as Theorem 2.8 in [3],[4] and Theorem 3.6 in [24], respectively.

Theorem 3.7. Define a sequence $s^{(k)} = \{s_n^{(k)}\}_{n \in \mathbb{N}}$ of elements of the space $bv(G)$ for every fixed $k \in \mathbb{N}$ by

$$s_n^{(k)} = \begin{cases} \frac{1}{v_n} \left(\frac{1}{u_k} - \frac{1}{u_{k-1}} \right) & , \quad (1 \leq k < n) \\ \frac{1}{u_n v_n} & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases}$$

Therefore, the sequence $\{s^{(k)}\}_{k \in \mathbb{N}}$ is a basis for the space $bv(G)$ and any $x \in bv(G)$ has a unique representation of the form

$$x = \sum_k (\Gamma x)_k s^{(k)}.$$

Theorem 3.8. We define the matrix $H = (h_{nk})$ as

$$(3.10) \quad h_{nk} = \begin{cases} \frac{1}{v_k} \left(\frac{1}{u_k} - \frac{1}{u_{k-1}} \right) a_n & , \quad (1 \leq k < n) \\ \frac{a_n}{u_n v_n} & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$, where $u, v \in \mathcal{U}, a = (a_k) \in \omega$. The α -dual of the space $bv(G)$ is the set

$$d_5 = \left\{ a = (a_k) \in \omega : \sup_{N \in \mathcal{F}} \sum_k |h_{nk}| < \infty \right\}.$$

Using the Theorem 2.3 and the matrix (3.10), we can give β - and γ -duals of the space $bv(G)$ as below:

Corollary 3.9. *Let $u, v \in U$ for all $k \in \mathbb{N}$. Then,*

$$\{bv(G)\}^\beta = \left\{ a = (a_k) \in \omega : \left\{ \frac{1}{v_k} \left(\frac{1}{u_k} - \frac{1}{u_{k-1}} \right) a_k \right\} \in \ell_1 \quad \text{and} \quad \left(\frac{a_n}{u_n v_n} \right) \in c \right\}$$

and

$$\{bv(G)\}^\gamma = \left\{ z = (z_k) \in \omega : \left\{ \frac{1}{v_k} \left(\frac{1}{u_k} - \frac{1}{u_{k-1}} \right) a_k \right\} \in \ell_1 \quad \text{and} \quad \left(\frac{a_n}{u_n v_n} \right) \in \ell_\infty \right\}.$$

Following theorems can be proved as Theorem 4.2 and 4.3 in [24].

Theorem 3.10. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$a_{nk} = \sum_{j=k}^{\infty} (u_j - u_{j-1}) v_k b_{nj} \quad \text{or} \quad b_{nk} = \frac{1}{v_k} \left(\frac{1}{u_k} - \frac{1}{u_{k-1}} \right) a_{nk}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then $A \in (bv(G) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{bv(G)\}^\beta$ for all $n \in \mathbb{N}$ and $B \in (\ell_1 : Y)$.

Theorem 3.11. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $T = (t_{nk})$ are connected with the relation*

$$t_{nk} = \sum_{j=0}^n (u_n - u_{n-1}) v_j a_{jk}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (Y : bv(G))$ if and only if $T \in (Y : \ell_1)$.

3.3. The space $bv(R)$. If we choose $u_n = 1/Q_n, v_k = q_k$ in the space $bv(G)$ defined by (3.7), we obtain the space $bv(R)$:

$$(3.11) \quad bv(R) = \left\{ x = (x_k) \in \omega : \sum_k \left| \sum_{j=0}^k (q_j/Q_k) x_j - \sum_{j=0}^{k-1} (q_j/Q_{k-1}) x_j \right| < \infty \right\}.$$

we can denote the space $bv(R)$ as $bv(R) = (bv)_{R^t} = (\ell_1)_{\Delta, R^t}$ as the spaces $bv(G)$, where $\Sigma = \sigma_{nk} = \Delta \cdot R^t$ defined by $\sigma_{nk} = (1/Q_n - 1/Q_{n-1})q_k$ ($1 \leq k < n$); $\sigma_{nk} = q_n/Q_n$ ($k = n$) and $\sigma_{nk} = 0$ ($k > n$) for all n .

The Σ -transform of the sequence $x = (x_k)$ defined by

$$(3.12) \quad y_k = \sum_{j=0}^{k-1} \left(\frac{1}{Q_k} - \frac{1}{Q_{k-1}} \right) q_j x_j + \frac{q_k}{Q_k} x_k.$$

Following theorems can be proved by Theorem 2.9, Theorem 2.7 in [1], respectively.

Theorem 3.12. *Define a sequence $p^{(k)} = \{p_n^{(k)}\}_{n \in \mathbb{N}}$ of elements of the space $bv(R)$ for every fixed $k \in \mathbb{N}$ by*

$$p_n^{(k)} = \begin{cases} \frac{(Q_k - Q_{k-1})}{q_n} & , \quad (1 \leq k < n) \\ \frac{q_n}{q_n} & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases}$$

Therefore, the sequence $\{p^{(k)}\}_{k \in \mathbb{N}}$ is a basis for the space $bv(R)$ and any $x \in bv(R)$ has a unique representation of the form

$$x = \sum_k (\Sigma x)_k p^{(k)}.$$

Theorem 3.13. We define the matrix $M = (m_{nk})$ as

$$(3.13) \quad m_{nk} = \begin{cases} \frac{(Q_n - Q_{n-1})}{q_k} a_n & , \quad (1 \leq k < n) \\ \frac{Q_n}{q_n} a_n & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. The α -dual of the space $bv(R)$ is the set

$$d_6 = \left\{ a = (a_k) \in \omega : \sup_{N \in \mathcal{F}} \sum_k |m_{nk}| < \infty \right\}.$$

Using the Theorem 2.3 and the matrix (3.13), we can give β - and γ -duals of the space $bv(R)$ as below:

Corollary 3.14.

$$\{bv(R)\}^\beta = \left\{ a = (a_k) \in \omega : \left\{ \frac{(Q_n - Q_{n-1})}{q_k} a_n \right\} \in \ell_1 \quad \text{and} \quad \left(\frac{Q_n}{q_n} a_n \right) \in c \right\}$$

and

$$\{bv(R)\}^\gamma = \left\{ z = (z_k) \in \omega : \left\{ \frac{(Q_n - Q_{n-1})}{q_k} a_n \right\} \in \ell_1 \quad \text{and} \quad \left(\frac{Q_n}{q_n} a_n \right) \in \ell_\infty \right\}.$$

Theorem 3.15. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation

$$a_{nk} = \sum_{j=k}^{\infty} \left(\frac{1}{Q_n} - \frac{1}{Q_{n-1}} \right) q_j b_{nj} \quad \text{or} \quad b_{nk} = \frac{(Q_n - Q_{n-1})}{q_k} a_{nk}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then $A \in (bv(R) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{bv(R)\}^\beta$ for all $n \in \mathbb{N}$ and $B \in (\ell_1 : Y)$.

Theorem 3.16. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $W = (w_{nk})$ are connected with the relation

$$w_{nk} = \sum_{j=0}^n \left(\frac{1}{Q_n} - \frac{1}{Q_{n-1}} \right) q_j a_{jk}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (Y : bv(R))$ if and only if $W \in (Y : \ell_1)$.

4. CONCLUSION

The matrix domain X_Δ for $X = \{\ell_\infty, c, c_0\}$ is called the *difference sequence spaces*, which was firstly defined and studied by Kizmaz [22]. If we choose $X = \ell_1$, the space $\ell_1(\Delta)$ is called the *space of all sequences of bounded variation* and denote by bv . The space bv_p consisting of all sequences whose differences are in the space ℓ_p . The space bv_p was introduced by Bařar and Altay [8]. More recently, the sequence spaces bv are studied in [8], [9], [19], [20], [21], [25], [30].

Several authors studied deal with the sequence spaces which obtained with the domain of the triangle matrices. It can be seen that the matrix domains for the matrix C_1 which known as Cesàro mean of order one in [7], [32], [34]; for the generalized weighted mean in [3], [4], [14], [15], [23], [24], [28], [31], [33]; for the Riesz mean in [1], [2], [10], [11], [13], [16], [17], [18], [26], [27]. Further, different works related to the matrix domain of the sequence spaces can be seen in [6].

In this work, we give well-known results related to some properties, dual spaces and matrix transformations of the sequence space bv and introduce the matrix domain of space bv with arbitrary triangle matrix A . Afterward, we choose the matrix A as Cesàro mean of order one, generalized weighted mean and Riesz mean and compute α -, β -, γ -duals of those spaces. And also, we characterize the matrix classes of the spaces $bv(C)$, $bv(G)$, $bv(R)$.

As a natural continuation of this paper, one can study the domain of different matrices instead of A . Additionally, sequence spaces in this paper can be defined by a index p for $1 \leq p < \infty$ and a bounded sequence of strictly positive real numbers (p_k) for $0 < p_k \leq 1$ and $1 < p_k < \infty$ and the concept almost convergence. And also it may be characterized several classes of matrix transformations between new sequence spaces in this work and sequence spaces which obtained with the domain of different matrices.

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